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A nonlinear Schrödinger equation with two symmetric point interactions in one dimension

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Abstract

We consider a time-dependent one-dimensional nonlinear Schrödinger equation with a symmetric double-well potential represented by two Dirac's δ . Among our results we give an explicit formula for the integral kernel of the unitary semigroup associated with the linear part of the Hamiltonian. Then we establish the corresponding Strichartz-type estimate and we prove local existence and uniqueness of the solution to the original nonlinear problem.

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1. Introduction

The presence of a certain symmetry in various physical systems is often linked to some symmetric double-well potential. Models with such kind of potentials are therefore of interest in many fields of research. One of the most important phenomena observed in these situations is the spontaneous symmetry breaking, which arises in a wide range of physical systems modelled by nonlinear equations with a double-well potential. For instance, in classical physics, it has been experimentally observed for laser beams in Kerr media and focusing nonlinearity [9, 19]. A spontaneous symmetry-breaking phenomenon may also arise in the so-called Bose Einstein condensates, where the effective double-well potential is formed by the combined effect of a parabolic-like trap and a periodic optical lattice [6, 14, 25]. From a theoretical point of view, nonlinear systems with a double-well potential have been recently studied in the semiclassical limit [7, 28] or in the limit of large distance between the two wells [23, 27].

In order to introduce a simplified model of nonlinear Schrödinger equations describing the basic features of systems with double-well potentials, we consider in this paper a

one-dimensional system where the two wells are represented by two Dirac’s δ potentials. In other words, we will deal with the nonlinear time-dependent Schrödinger equation

$$\begin{cases} i\partial_t \psi_t = H_\alpha \psi_t + v|\psi_t|^{2\mu} \psi_t \\ \psi_t(x)|_{t=0} = \psi_0(x), \end{cases} \quad (1.1)$$

where ψ_0 is the initial data, μ, v are positive numbers and $H_\alpha, \alpha \in \mathbb{R}$, denotes a one-dimensional Schrödinger operator with symmetric delta interactions of strength α and placed at the points $x = \pm a$, see section 2 below for a precise definition of H_α . Although the stationary states of the symmetry breaking bifurcation for equation (1.1) with cubic nonlinearity have been studied already by Jackson and Weinstein [22], much less is known about the time behaviour of the wavefunction for the fixed distance between the two wells.

The evolution equation (1.1) with one delta interaction has been considered by several authors [2, 15, 16, 18, 20, 21, 29, 33]. In [2], among other things, local existence and uniqueness of the solution of the associated nonlinear evolution equation is proved. The key point in the analysis of the corresponding nonlinear problem in [2] is a Strichartz-type estimate for the time propagator of the free Laplacian with one delta interaction restricted to the absolutely continuous part of the spectrum. Such a Strichartz-type inequality follows, in the model with one delta interaction, easily from the (quasi)explicit formula for the time propagator previously given by [3, 4, 17, 31].

Here we will deal with the situation when we have two delta interactions of the same strength. As in [2] the crucial ingredient is again a suitable Strichartz-type estimate. The main technical difficulty with respect to the case treated in [2] is the fact that we do not have an explicit expression for the associated time propagator at hand. Therefore, as a first step in the analysis of equation (1.1), we derive a formula for the integral kernel of the time propagator of H_α in the form of a series of certain special functions (see theorem 3.1). In the next step we establish the corresponding Strichartz-type estimate (theorem 3.5). With the help of this Strichartz estimate and standard tools of the analysis of nonlinear Schrödinger equations [8], we then prove the local existence and uniqueness of a solution of (1.1).

We would like to point out that our motivation was not only to obtain the decay estimate (3.3) for the evolution operator but also to give an explicit expression for the integral kernel of the evolution operator, which can be used for numerical simulations of equation (1.1) based on spectral splitting methods, see e.g. [29] for the numerical analysis of equation (1.1) with one single Dirac’s δ .

The text is organized as follows. After the preliminary section 2, where we briefly recall some basic properties of the operator H_α , we announce the main results of our paper (see section 3). Most of the proofs are presented in section 4. Some lengthy technical proofs are postponed to the appendices. Concerning the notation, we denote by $\|u\|_p$ the norm of a function u in $L^p(\mathbb{R})$. In the case $p = 2$ we will sometimes write $\|u\|$ instead of $\|u\|_2$.

2. Preliminaries

The operator H_α in $L^2(\mathbb{R})$ is associated with the closed quadratic form

$$Q[\psi] = \|\psi'\|^2 + \alpha(|\psi(a)|^2 + |\psi(-a)|^2), \quad \psi \in D(Q) = H^1(\mathbb{R}). \quad (2.1)$$

It is well known, see e.g. [5], that the functions from the domain of H_α satisfy the boundary conditions

$$\psi(\pm a + 0) = \psi(\pm a - 0) \quad (2.2)$$

$$\alpha\psi(\pm a + 0) = \psi'(\pm a + 0) - \psi'(\pm a - 0). \quad (2.3)$$

Then H_α acts on its domain $D(H_\alpha)$ as

$$H_\alpha \psi = -\psi'', \quad D(H_\alpha) = \{\psi \in H^2(\mathbb{R} \setminus \{\pm a\}) : \psi \text{ satisfies (2.2) and (2.3)}\}.$$

Note that, due to (2.2), the function $\psi \in D(H_\alpha)$ is continuous at $x = \pm a$ and therefore $D(H_\alpha)$ is a subspace of $H^1(\mathbb{R})$. Thus, in the following we denote by $\psi(\pm a)$ the limit (2.2). It is useful to recall some basic properties of the spectrum of H_α . The essential spectrum of H_α is purely absolutely continuous and coincides with the positive real axis:

$$\sigma_{\text{ess}}(H_\alpha) = \sigma_{\text{ac}}(H_\alpha) = [0, +\infty).$$

From the explicit form of the resolvent operator $(H_\alpha - z)^{-1}$, see [5, chapter II.2], it is straightforward to determine the discrete spectrum of H_α : if $\alpha \geq 0$, then the discrete spectrum of H_α is empty. For $\alpha < 0$ the discrete spectrum consists of negative eigenvalues E given by the implicit equation

$$(-2ik + \alpha)^2 = \alpha^2 e^{i4ka}, \quad k = \sqrt{E}, \quad \Im k \geq 0.$$

In particular, if $a \leq -\frac{1}{\alpha}$, then the discrete spectrum of H_α consists of only one eigenvalue $E_1(a, \alpha)$ defined by

$$E_1(a, \alpha) = -\frac{1}{4a^2} [W(-a\alpha e^{a\alpha}) - a\alpha]^2, \tag{2.4}$$

where $W(\cdot)$ is Lambert's special function defined by the equation $W(x)e^{W(x)} = x$, [10]. If $a > -\frac{1}{\alpha}$, then the discrete spectrum of H_α consists of two eigenvalues: $E_1(a, \alpha)$ and

$$E_2(a, \alpha) = -\frac{1}{4a^2} [W(+a\alpha e^{a\alpha}) - a\alpha]^2. \tag{2.5}$$

3. Main results

We first state a result which gives an explicit formula for the evolution operator associated with H_α in terms of a series of parabolic cylinder functions. We then make use of this formula to study the local existence and uniqueness of the nonlinear problem (1.1). In order to formulate our results we need to introduce some notation. Let P_c be the spectral projector of H_α on its absolute continuous spectrum. Moreover, define

$$\begin{aligned} z_n &:= z_n(x, y) = 4an + |x + a| + |y + a| \\ a_n &:= a_n(x, y, t) = \frac{z_n}{4\sqrt{t}} - \frac{\alpha\sqrt{t}}{4}, \quad b_n := b_n(x, y, t) = \frac{z_n}{4\sqrt{t}} + \frac{\alpha\sqrt{t}}{4}. \end{aligned}$$

Theorem 3.1. *Let $t > 0$ and assume that $a\alpha \neq -1$. Then the integral kernel of the operator $\exp(-it H_\alpha) P_c$ is given by*

$$U_\alpha(t; x, y) = \frac{1}{\sqrt{4\pi it}} e^{i|x-y|^2/4t} - \frac{1}{2\pi} \sum_{m=0}^{+\infty} [p_m(t; x, y) + p_m(t; -x, -y)], \tag{3.1}$$

where

$$\begin{aligned} p_m(t; x, y) &= (-1)^m r_{\frac{m}{2}}(t; x, (-1)^m y), \\ r_n(t; x, y) &= \text{sign}(\alpha) \sqrt{\frac{\pi}{2}} |\alpha|^{2n+1} \left(\frac{it}{2}\right)^n e^{i\frac{\pi}{4}} e^{i(a_n - b_n)^2} D[-2n - 1, -\text{sign}(\alpha)2i(a_n + ib_n)] \end{aligned} \tag{3.2}$$

and $D[n, z]$ denotes the parabolic cylinder function. Moreover, the series on the right-hand side of (3.1) converges for any $t > 0$ uniformly with respect to x and y .

Remark 3.2. Here we assume $a\alpha \neq -1$ for technical reasons. Note that for $a\alpha = -1$ the second eigenvalue E_2 of H_α emerges from the continuous spectrum.

Remark 3.3. The evolution operator e^{-itH_α} then has the kernel

$$\mathcal{V}_\alpha(t; x, y) = \Theta(-\alpha) e^{-itE_1} \bar{\varphi}_1(y) \varphi_1(x) + \Theta(-1 - a\alpha) e^{-itE_2} \bar{\varphi}_2(y) \varphi_2(x) + U_\alpha(t; x, y),$$

where φ_1 and φ_2 are the normalized eigenfunctions associated with the eigenvalues E_1 and E_2 and $\Theta(\cdot)$ denotes the Heaviside function.

Remark 3.4. Time evolution in the models involving one or more delta interactions similar to ours has been studied, apart from [2], in many other works (see e.g. [11–13]). For example in [11] a model of ionization is considered in which to the unperturbed Hamiltonian $-\frac{d^2}{dx^2} + g\delta(x)$, with $g > 0$, a perturbing time-dependent potential of the form $-g\eta(t)\delta(x)$, where $\eta(t)$ is a periodic function of time, is applied. It is then shown that the survival probability of the bound states of the unperturbed Hamiltonian tends to zero as $t \rightarrow \infty$ and the nature of this decay is discussed in detail. Similar survival probability is studied in [12] for a model where the perturbing potential is of the form $\eta(t)(\delta(x+a) - \delta(x-a))$ with $\eta(t) = r \sin(\omega t)$.

In our model the linear part of the Schrödinger equation corresponds to the Hamiltonian H_α with a time-independent potential represented by two symmetric delta interactions. Therefore, instead of looking at the time evolution of the (possible) bound states, we focus on the time evolution of the ‘continuous part’ of the operator, i.e. on the evolution operator restricted to the absolutely continuous part of the spectrum.

As a consequence of theorem 3.1, we obtain the following dispersion estimate.

Corollary 3.5. Assume that $a\alpha \neq -1$. Then there exists a constant C such that for any $u \in L^1(\mathbb{R})$ and any $t > 0$ it holds that

$$\|e^{-itH_\alpha} P_c u\|_\infty \leq Ct^{-\frac{1}{2}} \|u\|_1. \tag{3.3}$$

Remark 3.6. The same decay behaviour for the model with one δ interaction was obtained in [2]. Dispersion estimates for Schrödinger operators with regular potentials have been well studied in the literature (see for example [24, 32]). In [32] it was shown, under certain regularity assumptions on the corresponding potential, that the decay rate $t^{-1/2}$ of the unitary group e^{-itH} as an operator from $L^1(\mathbb{R})$ to $L^\infty(\mathbb{R})$ is typical for one-dimensional Schrödinger operators.

Although the assumptions of [24, 32] do not allow us to include directly the situation treated in our model, it is reasonable to expect that the dispersion estimate (3.3) could also be obtained by approximating the delta interactions by suitable sequences of regular potentials and applying the results of [32].

Our main result concerning the evolution problem (1.1) reads as follows.

Theorem 3.7. Suppose that the assumptions of theorem 3.1 are satisfied. If $\psi_0 \in H^1(\mathbb{R})$, then the problem (1.1) admits a unique local solution

$$\psi_t \in C((0, T), H^1(\mathbb{R})) \cap C^1((0, T), H^{-1}(\mathbb{R})) \tag{3.4}$$

for some $T > 0$. Moreover, if ψ_t is a solution of (1.1), then

$$\|\psi_t\| = \|\psi_0\|, \quad \mathcal{E}[\psi_t] = \mathcal{E}[\psi_0] \quad t \in (0, T), \tag{3.5}$$

where \mathcal{E} is the energy functional on $H^1(\mathbb{R})$ given by

$$\mathcal{E}[\psi] = \|\partial_x \psi\|^2 + \alpha[|\psi(a)|^2 + |\psi(-a)|^2] + \frac{\nu}{\mu + 1} \|\psi\|_{2\mu+2}^{2\mu+2}. \tag{3.6}$$

4. Proofs of the main results

We will need the explicit form of the integral kernel of the resolvent of H_α . According to [5, chapter II.2], we have

$$([H_\alpha - k^2]^{-1}\phi)(x) = \int_{\mathbb{R}} K_\alpha(x, y; k)\phi(y) dy, \quad \phi \in L^2(\mathbb{R}), \quad \Im k \geq 0,$$

where the integral kernel K_α is given by

$$K_\alpha(x, y; k) = K_0(x, y; k) + \sum_{j=1}^4 K_\alpha^j(x, y; k) \tag{4.1}$$

with

$$K_\alpha^j(x, y; k) = -(2k((2k + i\alpha)^2 + \alpha^2 e^{i4ka}))^{-1} L_\alpha^j(x, y; k), \quad K_0(x, y; k) = \frac{i}{2k} e^{ik|x-y|} \tag{4.2}$$

and

$$\begin{aligned} L_\alpha^1(x, y; k) &= -\alpha(2k + i\alpha) e^{ik|x+a|} e^{ik|y+a|}, & L_\alpha^4(x, y; k) &= L_\alpha^1(-x, -y; k) \\ L_\alpha^2(x, y; k) &= i\alpha^2 e^{2ika} e^{ik|x+a|} e^{ik|y-a|}, & L_\alpha^3(x, y; k) &= L_\alpha^2(-x, -y; k). \end{aligned} \tag{4.3}$$

4.1. Evolution operator of the linear equation

We split the proof of theorem 3.1 into several lemmata. We start by studying the properties of the function $U_\alpha(t; x, y)$ defined by

$$U_\alpha(t; x, y) = -\frac{i}{\pi} \int_{\mathbb{R}} k e^{-ik^2 t} K_\alpha(x, y; k) dk. \tag{4.4}$$

Note that

$$U_\alpha(t; x, y) = U_0(t; x, y) + \frac{i}{2\pi} \int_{\mathbb{R}} e^{-ik^2 t} \frac{f_\alpha(x, y; k)}{(2k + i\alpha)^2 + \alpha^2 e^{i4ka}} dk \tag{4.5}$$

with

$$f_\alpha(x, y; k) = \sum_{j=1}^4 L_\alpha^j(x, y; k), \quad U_0(t; x, y) = \frac{1}{\sqrt{4\pi it}} e^{i|x-y|^2/4t}.$$

Lemma 4.1. *For any $t > 0$ it holds that*

$$U_\alpha(t; x, y) = U_0(t; x, y) - \frac{1}{2\pi} \sum_{j=1}^4 U_j(t; x, y), \tag{4.6}$$

where

$$U_j(t; x, y) = -i \sum_{n=0}^{\infty} \int_{\mathbb{R}} e^{-ik^2 t} \frac{L_\alpha^j(x, y; k)}{(2k + i\alpha)^2} \left(-\frac{\alpha^2 e^{i4ka}}{(2k + i\alpha)^2} \right)^n dk. \tag{4.7}$$

Proof. By (4.5) we have

$$\begin{aligned} U_\alpha(t; x, y) - U_0(t; x, y) &= \frac{i}{2\pi} \int_{\mathbb{R}} e^{-ik^2 t} \frac{f_\alpha(x, y; k)}{(2k + i\alpha)^2 + \alpha^2 e^{i4ka}} dk \\ &= \frac{i}{2\pi} \int_{\mathbb{R}} e^{-ik^2 t} \sum_{n=0}^{\infty} \frac{f_\alpha(x, y; k)}{(2k + i\alpha)^2} \left(-\frac{\alpha^2 e^{i4ka}}{(2k + i\alpha)^2} \right)^n dk. \end{aligned} \tag{4.8}$$

Since

$$f_\alpha(x, y; k) = \mathcal{O}(k) \quad k \rightarrow 0,$$

for each x, y the series on the right-hand side of (4.8) converges uniformly with respect to k on \mathbb{R} . We can thus interchange the summation and the integration to get (4.7). \square

Lemma 4.2. For $t > 0$ and $\alpha \neq -1$ we have

$$U_1(t; x, y) = \sum_{n=0}^{\infty} r_n(t; x, y), \quad U_2(t; x, y) = \sum_{n=0}^{\infty} \hat{r}_{n+\frac{1}{2}}(t; x, y),$$

$$\hat{r}_n(t; x, y) = -r_n(t; x, -y).$$

Proof. By lemma 4.1

$$U_1(t; x, y) = \sum_{n=0}^{\infty} i^{2n+1} \alpha^{2n+1} \int_{\mathbb{R}} e^{-ik^2 t} \frac{e^{i4nka}}{(2k + i\alpha)^{2n+1}} e^{ik|x+a|} e^{ik|y+a|} dk$$

$$= \sum_{n=0}^{\infty} i^{2n+1} \alpha^{2n+1} e^{\frac{iz_n^2}{4t}} 2^{-(2n+1)} t^n \int_{\mathbb{R}} e^{-ir^2} \left(r + \frac{z_n}{2\sqrt{t}} + \frac{i\alpha\sqrt{t}}{2} \right)^{-(2n+1)} dr$$

$$= \sum_{n=0}^{\infty} i^{2n+1} \alpha^{2n+1} e^{\frac{iz_n^2}{4t}} 2^{-(2n+1)} t^n A_{2n+1} \left(\frac{z_n}{2\sqrt{t}}, \frac{\alpha\sqrt{t}}{2} \right)$$

$$= \sum_{n=0}^{\infty} \text{sign}(\alpha) \sqrt{\frac{\pi}{2}} \alpha^{2n+1} \left(\frac{it}{2} \right)^n e^{\frac{iz_n^2}{4t}} e^{-(a_n+ib_n)^2} D[-2n-1, -\text{sign}(\alpha)2i(a_n+ib_n)],$$

(4.9)

where we apply formula (A.6) (see appendix A). The calculation of $U_2(t; x, y)$ follows the same line. Let $\hat{z}_n := \hat{z}_n(x, y) = z_n(x, -y)$ and define

$$\hat{a}_n := \hat{a}_n(x, y, t) = \frac{\hat{z}_n}{4\sqrt{t}} - \frac{\alpha\sqrt{t}}{4} \quad \text{and} \quad \hat{b}_n := \hat{b}_n(x, y, t) = \frac{\hat{z}_n}{4\sqrt{t}} + \frac{\alpha\sqrt{t}}{4}.$$

As above we obtain

$$U_2(t; x, y) = - \sum_{n=0}^{\infty} i^{2n+2} \alpha^{2n+2} \exp(i(\hat{z}_{n+\frac{1}{2}})^2/4t) 2^{-(2n+2)} t^{n+\frac{1}{2}} A_{2n+2} \left(\frac{\hat{z}_{n+\frac{1}{2}}}{2\sqrt{t}}, \frac{\alpha\sqrt{t}}{2} \right)$$

$$= - \sum_{n=0}^{\infty} \text{sign}(\alpha) \sqrt{\frac{\pi}{2}} \alpha^{2n+2} \left(\frac{it}{2} \right)^{\frac{2n+1}{2}} \exp(i(\hat{z}_{n+\frac{1}{2}})^2/4t) e^{-(\hat{a}_{n+\frac{1}{2}}+i\hat{b}_{n+\frac{1}{2}})^2}$$

$$\times D[-2n-2, -\text{sign}(\alpha)2i(\hat{a}_{n+\frac{1}{2}}+i\hat{b}_{n+\frac{1}{2}})],$$

where we again applied formula (A.6). \square

We will also need a uniform estimate on the sequence $r_n(t; x, y)$ defined in (3.2).

Lemma 4.3. Let $t > 0$ be fixed and assume that $\alpha \neq -1$. Then there exists a constant C independent of t, x and y , such that for all $n \geq 1$ we have

$$|r_n(t; x, y)|, |\hat{r}_{n+\frac{1}{2}}(t; x, y)| \leq C^n t^{\frac{1}{2}} \frac{t^{2n} n^{-2n}}{(1+|x+a|+|y+a|)^{2n}}. \quad (4.10)$$

Proof. From the definition of a_n, b_n , lemma A.1, appendix A, and lemma 4.2, we get

$$|r_n(t; x, y)| \leq \sqrt{\frac{\pi}{2}} |\alpha|^{2n+1} \left(\frac{t}{2}\right)^n |I_n(t; x, y)|, \tag{4.11}$$

where

$$I_n(t; x, y) = \int_{\mathbb{R}} e^{-s^2} (s + \sqrt{2}a_n + i\sqrt{2}b_n)^{-2n-1} ds$$

Hence

$$|I_n(t; x, y)| \leq \sqrt{\pi} |\sqrt{2}b_n|^{-2n-1} \leq C \left(\frac{t}{z_n}\right)^n,$$

which implies (4.10). The estimate for $|r_{n+\frac{1}{2}}(t; x, y)|$ is completely analogous. □

Next we show that $U_\alpha(t; x, y)$ defines the kernel of the evolution operator associated with H_α .

Lemma 4.4. *Let $K_\alpha(x, y; k)$ be the kernel of the resolvent operator of H_α and let P_c be the spectral projection of H_α on $[0, \infty)$. Then for any test function $\phi \in C_0^\infty(\mathbb{R})$ we have*

$$(e^{-itH_\alpha} P_c \phi)(x) = \int_{\mathbb{R}} U_\alpha(t; x, y) \phi(y) dy. \tag{4.12}$$

Proof. Fix $\phi \in C_0^\infty(\mathbb{R})$. By [30, theorem 3.1] it holds that

$$\lim_{\epsilon \rightarrow 0^+} e^{-(it+\epsilon)H_\alpha} \phi = e^{-itH_\alpha} \phi. \tag{4.13}$$

Since the absolute continuous spectrum of H_α is the interval $[0, \infty)$, the spectral theorem gives

$$e^{-(it+\epsilon)H_\alpha} P_c = \int_0^\infty e^{-(it+\epsilon)\lambda} dP_\lambda,$$

where P_λ is the spectral projector of H_α on the interval $[0, \lambda]$. By Stone's formula, see e.g. [26, theorem VII.13], we have that for any $\lambda \geq 0$

$$P([0, \lambda]) = \lim_{\delta \rightarrow 0^+} \frac{1}{2\pi i} \int_0^\lambda \{ [H_\alpha - (z + i\delta)]^{-1} - [H_\alpha - (z - i\delta)]^{-1} \} dz.$$

Hence

$$\begin{aligned} e^{-(it+\epsilon)H_\alpha} P_c \phi &= \frac{1}{2\pi i} \int_0^\infty e^{-(it+\epsilon)z} \lim_{\delta \rightarrow 0^+} \int_{\mathbb{R}} \{ K_\alpha(x, y; \sqrt{z + i\delta}) - K_\alpha(x, y; \sqrt{z - i\delta}) \} \phi(y) dy dz \\ &= \frac{1}{\pi i} \int_{-\infty}^0 e^{-(it+\epsilon)k^2} \lim_{\delta \rightarrow 0^+} \int_{\mathbb{R}} k K_\alpha(x, y; \sqrt{k^2 - i\delta}) \phi(y) dy dk \\ &\quad + \frac{1}{\pi i} \int_0^\infty e^{-(it+\epsilon)k^2} \lim_{\delta \rightarrow 0^+} \int_{\mathbb{R}} k K_\alpha(x, y; \sqrt{k^2 + i\delta}) \phi(y) dy dk, \end{aligned} \tag{4.14}$$

where we have substituted $k = \sqrt{z}$ in the integral containing the kernel $K_\alpha(x, y; \sqrt{z + i\delta})$ and $k = -\sqrt{z}$ in the integral containing the kernel $K_\alpha(x, y; \sqrt{z - i\delta})$. Since $\phi \in C_0^\infty(\mathbb{R})$ and $|k K_\alpha(x, y; \sqrt{k^2 \pm i\delta})|$ is uniformly bounded for $\delta \geq 0$ small enough, we can exchange the limit and integration in (4.14). Note also that

$$\lim_{\delta \rightarrow 0^+} k K_\alpha(x, y; \sqrt{k^2 \pm i\delta}) = k K_\alpha(x, y; \pm |k|) \quad \forall k \neq 0$$

and that $kK_\alpha(x, y; k) = \mathcal{O}(1)$ as $k \rightarrow 0$. From the Fubini theorem we then obtain

$$\begin{aligned} (e^{-(it+\epsilon)H_\alpha} P_\epsilon \phi)(x) &= \frac{1}{\pi i} \int_{\mathbb{R}} \phi(y) \int_{\mathbb{R}} kK_\alpha(x, y, k) e^{-k^2(it+\epsilon)} dk dy \\ &= \int_{\mathbb{R}} U_\alpha(t - i\epsilon; x, y) \phi(y) dy. \end{aligned} \tag{4.15}$$

Passing to the limit $\epsilon \rightarrow 0+$ in (4.15) gives

$$(e^{-itH_\alpha} P_\epsilon \phi)(x) = \lim_{\epsilon \rightarrow 0+} \int_{\mathbb{R}} U_\alpha(t - i\epsilon; x, y) \phi(y) dy \tag{4.16}$$

(see (4.13)). By (4.5) it can be directly verified that for any $x, y \in \mathbb{R}$ it holds that

$$\lim_{\epsilon \rightarrow 0+} U_\alpha(t - i\epsilon; x, y) = U_\alpha(t; x, y). \tag{4.17}$$

Indeed, since $U_0(t; x, y)$ is continuous in $t \in \mathbb{C} \setminus \{0\}$, it follows that

$$U_\alpha(t; x, y) - U_\alpha(t - i\epsilon; x, y) = -\frac{i}{\pi} \int_{\mathbb{R}} e^{-ik^2t} \frac{f_\alpha(x, y; k)(1 - e^{-\epsilon k^2})}{(2k + i\alpha)^2 + \alpha^2 e^{i4ka}} dk + o(1) \tag{4.18}$$

as $\epsilon \rightarrow 0+$. To show that the first term on the right-hand side of (4.18) converges to zero, we observe that

$$\lim_{\epsilon \rightarrow 0+} \int_{-R}^R e^{-ik^2t} \frac{f_\alpha(x, y; k)(1 - e^{-\epsilon k^2})}{(2k + i\alpha)^2 + \alpha^2 e^{i4ka}} dk = 0 \tag{4.19}$$

for any $R > 0$. On the other hand, a direct calculation shows that

$$\frac{f_\alpha(x, y; k)}{(2k + i\alpha)^2 + \alpha^2 e^{i4ka}} = -\frac{\alpha}{2k} (e^{ik|x+a|} e^{ik|y+a|} + e^{ik|x-a|} e^{ik|y-a|}) + \mathcal{O}(k^{-2}),$$

where the term $\mathcal{O}(k^{-2})$ depends also on x, y and α . From the asymptotic relation

$$\int_R^\infty e^{-zk^2} \frac{1}{k} dk = \mathcal{O}\left(\frac{e^{-zR^2}}{zR^2}\right), \quad R \rightarrow \infty, \quad |\arg z| < \pi$$

(see e.g. [1, chapter 5]), we thus deduce that

$$\int_R^\infty e^{-zk^2} \frac{f_\alpha(x, y; k)(1 - e^{-\epsilon k^2})}{(2k + i\alpha)^2 + \alpha^2 e^{i4ka}} dk = \mathcal{O}(R^{-2}), \quad R \rightarrow \infty,$$

uniformly with respect to $\epsilon \geq 0$. The integral corresponding to $k \in (-\infty, -R)$ is treated in the same way. This together with (4.19) implies (4.17). It follows that for any $x \in \mathbb{R}$ the function $|U_\alpha(t - i\epsilon; x, y)|$ is uniformly bounded for ϵ nonnegative and small enough and y on compact subsets of \mathbb{R} . Hence we can interchange the limit with the integral in (4.16) to conclude the proof. \square

Proof of theorem 3.1. Since

$$U_3(t; x, y) = U_2(t; -x, -y), \quad U_4(t; x, y) = U_1(t; -x, -y), \tag{4.20}$$

the statement of the theorem follows from lemmata 4.1, 4.2, 4.3 and 4.4. \square

4.2. Proof of corollary 3.5

We need two auxiliary lemmata.

Lemma 4.5. *Let $\alpha \neq -1$. For any $T > 0$, there exists a constant C_T such that*

$$\sup_{x,y \in \mathbb{R}} |U_\alpha(t; x, y)| \leq C_T t^{-\frac{1}{2}}, \quad t \in [0, T].$$

Proof. Clearly

$$|U_0(t; x, y)| \leq \frac{1}{\sqrt{4\pi t}}.$$

It thus remains to estimate $U_j(t; x, y)$, $j = 1, 2, 3, 4$. By (4.10) we have

$$\left| \sum_{n=1}^{\infty} r_{n+\frac{1}{2}}(t; x, y) \right|, \left| \sum_{n=1}^{\infty} r_n(t; x, y) \right| \leq C' t^{\frac{1}{2}}, \quad t \in [0, T],$$

where the constant C' depends only on T . On the other hand, for $n = 0$, it follows directly from the definition that

$$|r_0(t; x, y)|, |r_{\frac{1}{2}}(t; x, y)| \leq C,$$

where C is a real number. □

An analogous upper bound on $U_\alpha(x, y; t)$ for large t is given in the following lemma whose proof we postpone to appendix B.

Lemma 4.6. *There exists a constant C such that for all t large enough, it holds that*

$$\sup_{x,y \in \mathbb{R}} |\sqrt{t} U_\alpha(x, y; t)| \leq C. \tag{4.21}$$

Proof of corollary 3.5. By lemmata 4.5 and 4.6, we have

$$\sup_{x,y \in \mathbb{R}} |U_\alpha(t; x, y)| \leq C t^{-\frac{1}{2}}$$

for all $t > 0$ and some constant C independent of t . Hence

$$\|e^{-itH_\alpha} P_c \phi\|_\infty \leq C t^{-\frac{1}{2}} \|\phi\|_1 \quad \forall \phi \in C_0^\infty(\mathbb{R}).$$

By density, this inequality extends to all $u \in L^1(\mathbb{R})$. □

As a consequence we obtain the following Strichartz-type estimate.

Corollary 4.7. *Let $r \geq 2$ and $q = \frac{4r}{r-2}$. Then for any $T > 0$ there exists a constant C such that*

$$\|e^{-itH_\alpha} P_c u\|_{L^q((0,T), L^r(\mathbb{R}))} \leq C \|u\|_2 \quad \forall u \in L^2(\mathbb{R}). \tag{4.22}$$

Proof. Let $0 < t < T$. In view of (3.3) and the obvious inequality

$$\|e^{-itH_\alpha} P_c u\|_2 \leq \|u\|_2,$$

the Riesz–Thorin interpolation theorem implies that

$$\|e^{-itH_\alpha} P_c u\|_p \leq C t^{\frac{1}{p} - \frac{1}{2}} \|u\|_r, \quad \frac{1}{p} + \frac{1}{r} = 1 \tag{4.23}$$

for some C and $p \in [2, \infty]$. Since H_α is a self-adjoint, (4.22) follows from (4.23) and [8, theorem 2.7.1]. □

4.3. Proof of theorem 3.7

In order to study the well posedness of the nonlinear equation (1.1), we follow the strategy adopted in [2]. To this end we define the operator T_α by

$$T_\alpha = \begin{cases} H_\alpha & \alpha \geq 0 \\ H_\alpha - E_1(a, \alpha) & \alpha < 0. \end{cases}$$

The key point, apart from the Strichartz estimate, is to prove the following technical result (see also [8, section 3.7]).

Proposition 4.8. *Let $p \in [2, \infty]$. Then the domain $D(T_\alpha)$ of T_α is embedded in $L^p(\mathbb{R})$. Moreover, there exists a constant C such that for any $\varepsilon < 0$ and any $u \in L^p(\mathbb{R})$ it holds that*

$$\|(\varepsilon T_\alpha - 1)^{-1}u\|_p \leq C\|u\|_p. \tag{4.24}$$

Proof. Consider first the case $\alpha > 0$. Since $H^1(\mathbb{R})$ is continuously embedded in $L^p(\mathbb{R})$ for any $p \in [2, \infty]$, the embedding $D(T_\alpha) = D(H_\alpha) \hookrightarrow L^p(\mathbb{R})$ follows in view of the fact that $D(H_\alpha)$ is a subspace of $H^1(\mathbb{R})$. To prove (4.24) we note that $(\varepsilon T_\alpha - 1)^{-1} = (\varepsilon H_\alpha - 1)^{-1}$ is an integral operator with the kernel

$$\varepsilon^{-1}K_\alpha(x, y; i\lambda) = \varepsilon^{-1}K_0(x, y; i\lambda) + \varepsilon^{-1} \sum_{j=1}^4 K_\alpha^j(x, y; i\lambda), \quad \lambda = -i\varepsilon^{-\frac{1}{2}} > 0.$$

First we observe that

$$\varepsilon^{-1} \int_{\mathbb{R}} K_0(x, y; i\lambda)u(y) dy = \frac{1}{2\lambda\varepsilon} (e^{-\lambda|\cdot|} * u)(x).$$

By the Young inequality we thus get

$$\left\| \varepsilon^{-1} \int_{\mathbb{R}} K_0(x, y; i\lambda)u(y) dy \right\|_p \leq \frac{1}{2\lambda|\varepsilon|} \|e^{-\lambda|\cdot|}\|_1 \|u\|_p = \|u\|_p.$$

To control the other terms in $K_\alpha(x, y; i\lambda)$, we write

$$K_\alpha^1(x, y; i\lambda) + K_\alpha^2(x, y; i\lambda) = \frac{-\alpha e^{-\lambda|x+a|} e^{-\lambda|y+a|}}{(2\lambda + \alpha)^2 - \alpha^2 e^{-4\lambda a}} + \begin{cases} \frac{\alpha^2 e^{-\lambda|x+a|} e^{-\lambda|y+a|}}{2\lambda((2\lambda + \alpha)^2 - \alpha^2 e^{-4\lambda a})} (1 - e^{-2\lambda a} e^{-\lambda(|y-a|-|y+a|)}) & y < 0 \\ \frac{\alpha^2 e^{-\lambda|x+a|} e^{-\lambda|y-a|}}{2\lambda((2\lambda + \alpha)^2 - \alpha^2 e^{-4\lambda a})} (e^{-\lambda(|y+a|-|y-a|)} - e^{-2\lambda a}) & 0 \leq y \end{cases} \tag{4.25}$$

(see equation (4.3)). It is easily seen that the function $\lambda/((2\lambda + \alpha)^2 - \alpha^2 e^{-4\lambda a})$ is bounded for $\lambda \in (0, \infty)$. On the other hand, for $y < 0$ we have $0 \leq |y - a| - |y + a| \leq 2a$. Hence for some constant $C > 0$, independent of $y < 0$, it holds that

$$\sup_{\lambda > 0} \frac{1 - e^{-2\lambda a} e^{-\lambda(|y-a|-|y+a|)}}{(2\lambda + \alpha)^2 - \alpha^2 e^{-4\lambda a}} \leq \sup_{\lambda > 0} \frac{1 - e^{-4\lambda a}}{(2\lambda + \alpha)^2 - \alpha^2 e^{-4\lambda a}} \leq C.$$

The last term in (4.25), which corresponds to $y \geq 0$, is estimated in the same way. We can thus conclude that there exists a constant $C_{\alpha,a}$, independent of x, y and λ , such that

$$\left| K_\alpha^1(x, y; i\lambda) + K_\alpha^2(x, y; i\lambda) \right| \leq C_{\alpha,a} \frac{1}{\lambda} e^{-\lambda|x+a|} (e^{-\lambda|y+a|} + e^{-\lambda||y|-a|}). \tag{4.26}$$

From the Hölder and Minkowski inequality we then get

$$\begin{aligned} & \left\| \int_{\mathbb{R}} (K_{\alpha}^1(x, y; i\lambda) + K_{\alpha}^2(x, y; i\lambda))u(y) dy \right\|_p \\ & \leq \frac{C_{\alpha,a}}{\lambda} \|e^{-\lambda|\cdot|}\|_p \left\| \int_{\mathbb{R}} (e^{-\lambda|y+a|} + e^{-\lambda||y|-a|})|u(y)| dy \right\|_{\infty} \\ & \leq \frac{C_{\alpha,a}}{\lambda} \|e^{-\lambda|\cdot|}\|_p \|e^{-\lambda|y+a|} + e^{-\lambda||y|-a|}\|_r \|u\|_p \\ & \leq \frac{C_{\alpha,a}}{\lambda^2} 2^{2+\frac{1}{r}} p^{-\frac{1}{p}} r^{-\frac{1}{r}} \|u\|_p, \end{aligned} \tag{4.27}$$

where $\frac{1}{p} + \frac{1}{r} = 1$. Since $K_{\alpha}^3(x, y; i\lambda) + K_{\alpha}^4(x, y; i\lambda) = K_{\alpha}^1(-x, -y; i\lambda) + K_{\alpha}^2(-x, -y; i\lambda)$, we arrive at

$$\frac{1}{|\varepsilon|} \sum_{j=1}^4 \left\| \int_{\mathbb{R}} K_{\alpha}^j(x, y; i\lambda)u(y) dy \right\|_p \leq \text{const} \|u\|_p. \tag{4.28}$$

This completes the proof in the case $\alpha > 0$.

For $\alpha \leq 0$ we redefine the parameter λ in the following way:

$$\lambda = -i\sqrt{\varepsilon^{-1} + E_1(a, \alpha)}, \quad \lambda > \sqrt{|E_1(a, \alpha)|},$$

where $E_1(a, \alpha) < 0$ is the lowest eigenvalue of H_{α} defined in (2.4). Then the integral kernel of $(\varepsilon T_{\alpha} - 1)^{-1}$ is again equal to $\varepsilon^{-1} K_{\alpha}(x, y; i\lambda)$. Moreover, there exists a constant $C_{\alpha,a}^1$, such that

$$|\varepsilon^{-1} K_{\alpha}^1(x, y; i\lambda)| \leq \frac{C_{\alpha,a}^1}{\lambda|\varepsilon||\lambda - \sqrt{|E_1(a, \alpha)|}} e^{-\lambda|x+a|} e^{-\lambda|y+a|},$$

and similarly for $K_{\alpha}^j(x, y; i\lambda)$, $j = 2, 3, 4$. The Hölder inequality applied in the same way as in (4.27) then gives

$$\frac{1}{|\varepsilon|} \sum_{j=1}^4 \left\| \int_{\mathbb{R}} K_{\alpha}^j(x, y; i\lambda)u(y) dy \right\|_p \leq \frac{\text{const}}{\lambda^2|\varepsilon||\lambda - \sqrt{|E_1(a, \alpha)|}} \|u\|_p.$$

Since

$$\lambda^2|\varepsilon||\lambda - \sqrt{|E_1(a, \alpha)|}| \geq \sqrt{|E_1(a, \alpha)|} \quad \forall \lambda > \sqrt{|E_1(a, \alpha)|},$$

the proof is completed. □

Proof of theorem 3.7. The Strichartz estimate (4.22) and a straightforward application of the proof of [8, proposition 4.2.1] imply the uniqueness, in $H^1(\mathbb{R})$, of a solution to the integral equation

$$\varphi_t = e^{-itT_{\alpha}} \varphi_0 - i\nu \int_0^t e^{i(s-t)T_{\alpha}} |\varphi_s|^2 \varphi_s ds. \tag{4.29}$$

Moreover, by proposition 4.8 and [8, theorems 3.7.1], we get the local existence of the solution to (4.29) and the conservation of energy and charge associated with this equation:

$$\mathcal{E}[\varphi_t] + \Theta(-\alpha)E_1(a, \alpha)\|\varphi_t\|^2 = \mathcal{E}[\varphi_0] + \Theta(-\alpha)E_1(a, \alpha)\|\varphi_0\|^2, \quad \|\varphi_t\|^2 = \|\varphi_0\|^2 \tag{4.30}$$

for $t \in (0, T)$. This immediately implies the local existence and uniqueness of the solution of

$$\psi_t = e^{-itH_{\alpha}} \psi_0 - i\nu \int_0^t e^{i(s-t)H_{\alpha}} |\psi_s|^2 \psi_s ds \tag{4.31}$$

by setting $\psi_t = e^{iE_1(a, \alpha)t} \varphi_t$ if $\alpha < 0$ and $\psi_t = \varphi_t$ otherwise. Since (4.31) is equivalent to (1.1), this proves the first part of the statement. The conservation of the charge and energy, equation (3.5), is a direct consequence of (4.30). □

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Appendix A.

Lemma A.1. *Let $n \in \mathbb{N}$, $\gamma > 0$, $\delta \neq 0$, and consider the following integral:*

$$A_{2n+1}(\gamma, \delta) = \int_{\mathbb{R}} e^{-ir^2} (r + \gamma + i\delta)^{-2n-1} dr. \tag{A.1}$$

(i) *If $\delta > 0$:*

$$A_{2n+1} = -i(-2i)^n \sqrt{2\pi} e^{-i(\gamma+i\delta)^2/2} D[-2n-1, (1-i)(\gamma+i\delta)]. \tag{A.2}$$

(ii) *If $\delta < 0$:*

$$A_{2n+1} = -i(-2i)^n \sqrt{2\pi} e^{-i(\gamma+i\delta)^2/2} D[-2n-1, (1-i)(\gamma+i\delta)] + i^n \frac{2\pi i}{(2n)!} e^{-i(\gamma+i\delta)^2} H_{2n}(-\sqrt{i}(\gamma+i\delta)), \tag{A.3}$$

where $D(n, z)$ is the parabolic cylinder function and where H_m is the Hermite's polynomial of degree m .

Proof. Let $a = -\gamma - i\delta$, where $\delta > 0$ and $z = r - a$, then the integral (A.1) takes the form

$$A_{2n+1} = e^{-ia^2} \int_{\mathbb{R}+i\delta} e^{-iz^2-2iza} z^{-2n-1} dz. \tag{A.4}$$

By means of the change of variable $s = e^{-i\pi/4}z$ and by the Cauchy theorem we then get

$$A_{2n+1} = e^{-ia^2} e^{-in\pi/2} \int_{(\ell)} e^{s^2-2ie^{i\pi/4}as} s^{-2n-1} ds = e^{-ia^2} e^{-in\pi/2} \int_{-(\epsilon)} e^{s^2-2ie^{i\pi/4}as} s^{-2n-1} ds,$$

where (ℓ) is the complex path $\{\zeta \in \mathbb{C} : \zeta = (x + i\delta) e^{-i\pi/4}, x \in \mathbb{R}\}$ and where (ϵ) is the complex path defined in figure 19.3 in [1]. Finally, we set $q = \sqrt{2}s$ and apply formula (19.5.4) by [1] to obtain

$$A_{2n+1} = -e^{-ia^2} (-i)^n 2^n \int_{(\epsilon)} e^{q^2/2-(\sqrt{2}ie^{i\pi/4}a)q} q^{-2n-1} dq = -e^{-ia^2} (-2i)^n \sqrt{2\pi} i e^{-(\sqrt{2}ie^{i\pi/4}a)^2/4} D[-2n-1, \sqrt{2}i e^{i\pi/4}a].$$

In the case $\delta < 0$ the integral (A.4) can be written as

$$A_{2n+1} = e^{-ia^2} \int_{\mathbb{R}-i|\delta|} e^{-iz^2-2iza} z^{-2n-1} dz = e^{-ia^2} \int_{\mathbb{R}+i|\delta|} e^{-iz^2-2iza} z^{-2n-1} dz + 2\pi i e^{-ia^2} \text{Res}[e^{-iz^2-2iza} z^{-2n-1}, z = 0],$$

where the integral is computed as the case of $\delta > 0$ and where the residue is equal to

$$\begin{aligned} \text{Res}[e^{-ir^2} (r - a)^{-2n-1}, r = a] &= \frac{1}{(2n)!} \frac{d^{2n}}{dr^{2n}} (e^{-ir^2}) \Big|_{r=a} = \frac{i^n}{(2n)!} \frac{d^{2n}}{ds^{2n}} (e^{-s^2}) \Big|_{s=\sqrt{ia}} \\ &= \frac{i^n}{(2n)!} e^{-ia^2} H_{2n}(\sqrt{ia}). \end{aligned} \quad \square$$

Remark A.2. By formulae (19.4.6) and (19.13.1) of [1] equations (A.2) and (A.3) can be written as

$$A_{2n+1} = -\text{sign}(\delta)i(-2i)^n \sqrt{2\pi} e^{-i(\gamma+i\delta)^2/2} D[-2n-1, \text{sign}(\delta)(1-i)(\gamma+i\delta)]. \tag{A.5}$$

Remark A.3. If we set

$$\chi = \frac{\gamma - \delta}{2} \quad \text{and} \quad \eta = \frac{\gamma + \delta}{2}, \quad \text{then}$$

$$A_{2n+1} = -\text{sign}(\delta)i(-2i)^n \sqrt{2\pi} e^{-(\chi+i\eta)^2} D[-2n-1, -\text{sign}(\delta)2i(\chi+i\eta)]. \tag{A.6}$$

Appendix B. Proof of lemma 4.6

Since $|U_0(t; x, y)| = 1/2\sqrt{\pi t}$, it suffices to consider $U_j(t; x, y)$, $j = 1, \dots, 4$. Let

$$f_\alpha(x, y; k) = \sum_{j=1}^4 L_\alpha^j(x, y; k) = 2\alpha k q(x, y; k) + i\alpha^2 p(x, y; k),$$

with

$$q(x, y; k) = -e^{ik|x+a|} e^{ik|y+a|} + e^{ik|x-a|} e^{ik|y-a|}, \quad p(x, y; k) = h(x, y; k) + h(-x, -y; k),$$

$$h(x, y; k) = e^{ik|x+a|} [e^{i2ka} e^{ik|y-a|} - e^{ik|y+a|}],$$

and define

$$r_\alpha(k) = \left(1 + \alpha \frac{e^{i4ka} - 1}{4ik} \right) 4i\alpha + 4k.$$

With this notation the statement of the lemma will follow, if we prove

Lemma B.1. *Let*

$$V_\alpha^t(x, y) = \int_{\mathbb{R}} e^{-ik^2 t} \frac{2\alpha q(x, y; k)}{r_\alpha(k)} dk, \quad W_\alpha^t(x, y) = \int_{\mathbb{R}} e^{-ik^2 t} \frac{i\alpha^2 p(x, y; k)}{kr_\alpha(k)} dk.$$

Then there exists a constant C such that for t large enough

$$\sup_{x,y} |V_\alpha^t(x, y) + W_\alpha^t(x, y)| \leq Ct^{-1/2}. \tag{B.1}$$

Proof. By means of the change of variable

$$\zeta = k\sqrt{t} - \frac{z}{2\sqrt{t}}, \quad z = |x+a| + |y+a|,$$

we get

$$|V_\alpha^t(x, y)| \leq \frac{C}{\sqrt{t}} \left| \int_{\mathbb{R}} e^{-i\zeta^2} \frac{1}{r_\alpha(\zeta/\sqrt{t} + z/2t)} d\zeta \right|.$$

We can estimate the above integral with the help of the Cauchy theorem for analytic functions replacing the domain of integration with the path

$$\gamma = \left\{ w \in \mathbb{C} : w = \zeta - \delta \frac{i\zeta}{\sqrt{1+\zeta^2}}, \delta > 0, \zeta \in \mathbb{R} \right\}.$$

Since the function $r_\alpha(k)$ has at most two zeros, both of them having the imaginary part strictly positive, it follows that for t large enough $|r_\alpha(w)|$ is bounded from below by some positive

constant in the strip $|\Im w| \leq \delta$. A direct calculation shows that for t large enough and some $C_\delta < \infty$ we have

$$\left| \int_{\mathbb{R}} e^{-i\zeta^2} \frac{1}{r_\alpha(\zeta/\sqrt{t} + z/2t)} d\zeta \right| \leq C \int_{\mathbb{R}} e^{-\frac{2\delta\zeta^2}{\sqrt{t+\zeta^2}}} d\zeta \leq C_\delta,$$

which yields the desired estimate for V_α^t :

$$\sup_{x,y} |V_\alpha^t(x, y)| \leq Ct^{-1/2}. \tag{B.2}$$

To prove the analogue of (B.2) for W_α^t , we first observe that

$$h(x, y; k) = 0 \quad \text{if } y \geq a.$$

We therefore define

$$\begin{aligned} h_1(x, y; k) &= h(x, y; k) = e^{ik|x+a|} e^{ik|y-a|} 2i \sin(2ka) & \text{if } y \leq -a \\ h_2(x, y; k) &= h(x, y; k) = e^{ik|x+a|} [e^{i2ka} e^{ik|y-a|} - e^{ik|y+a|}] = k e^{ik|x+a|} S(y; k) & \text{if } |y| \leq a, \end{aligned}$$

and consider separately the integrals

$$I_j(x, y, t) = \int_{\mathbb{R}} e^{-ik^2t} \frac{h_j(x, y; k)}{kr_\alpha(k)} dk, \quad j = 1, 2.$$

For the first integral we have

$$\begin{aligned} I_1(x, y, t) &= \int_{\mathbb{R}} e^{-ik^2t} \frac{h_1(x, y; k)}{kr_\alpha(k)} dk = \int_{\mathbb{R}} e^{-ik^2t} \frac{e^{ikz} 2i \sin(2ka)}{kr_\alpha(k)} dk \\ &= \frac{4ia}{\sqrt{t}} e^{iz^2/4t} \int_{\mathbb{R}} e^{i\zeta^2} \frac{\sin[2a(\zeta/\sqrt{t} + z/2t)]}{2a(\zeta/\sqrt{t} + z/2t)} \frac{d\zeta}{r_\alpha(\zeta/\sqrt{t} + z/2t)}, \end{aligned}$$

where $z = |x + a| + |y - a|$. Note that for any $\delta > 0$ there exists a constant C_δ such that

$$\left| \frac{\sin[2a(\zeta/\sqrt{t} + z/2t)]}{2a(\zeta/\sqrt{t} + z/2t)} \right| \leq C_\delta \quad |\Im \zeta| \leq \delta.$$

We can thus mimic the argument of the proof of lemma B.1 to arrive at the estimate

$$|I_1(x, y, t)| \leq C/\sqrt{t},$$

which holds for all t large enough uniformly in x and y . The same arguments apply to the integral

$$\begin{aligned} I_2(x, y, t) &= \int_{\mathbb{R}} e^{-ik^2t} \frac{h_2(x, y; k)}{kr_\alpha(k)} dk = \int_{\mathbb{R}} e^{-ik^2t} e^{ik|x+a|} \frac{S(y; k)}{r_\alpha(k)} dk \\ &= \frac{1}{\sqrt{t}} e^{i|x+a|^2/4t} \int_{\mathbb{R}} e^{i\zeta^2} \frac{S(y; \zeta/\sqrt{t} + |x+a|/2t)}{r_\alpha(\zeta/\sqrt{t} + |x+a|/2t)} d\zeta \end{aligned}$$

since

$$|S(y; w)| \leq C, \quad \forall y \in [-a, a], \quad |\Im w| \leq \delta.$$

Hence we can conclude that for t large enough

$$\sup_{x,y} |W_\alpha^t(x, y)| \leq Ct^{-1/2}. \tag{B.3}$$

□

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